① It follows from standard linear algebra. Note that each matrix in $SL_2(k)$ is conjugate to a matrix of the form $\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$, $\lambda \neq \pm 1$

or $\begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}$,

since the determinant of a matrix in $SL_2(k)$ is 1.

Also note that the trace is invariant in conjugacy classes.

② (i) Let $A_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \neq \pm 1$.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Cent(A_\lambda)$.

Then $g A_\lambda g^{-1} = A_\lambda$.

But $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \lambda ad - \lambda^{-1} bc & -\lambda ab + \lambda^{-1} ab \\ \lambda cd - \lambda^{-1} cd & -\lambda bc + \lambda^{-1} ad \end{pmatrix}$

$= \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$.

Since $\lambda \neq \pm 1$, this implies $ab = cd = 0$.

If $b = 0$, then $c = 0$ and $ad = 1$. Hence $g \in T$.

If $a = 0 \Rightarrow d = 0$, so $bc = -1$.

Here $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, contradiction.

This shows $\mathrm{Cent}(A_\lambda) = T$.

Now if g semisimple, then $g = \gamma A_\lambda \gamma^{-1}$, for some $\gamma \in SL_2(k)$, $\lambda \neq \pm 1$

$$\begin{aligned} \sigma \in \mathrm{Cent}(g) &\Leftrightarrow \sigma g \sigma^{-1} = g \\ &\Leftrightarrow \sigma \gamma A_\lambda \gamma^{-1} \sigma^{-1} = \gamma A_\lambda \gamma^{-1} \\ &\Leftrightarrow \gamma^{-1} \sigma \gamma \in \mathrm{Cent}(A_\lambda) = T \\ &\Leftrightarrow \sigma \in \gamma T \gamma^{-1}. \end{aligned}$$

(ii) If $N_x^\varepsilon = \begin{pmatrix} \varepsilon & x \\ 0 & \varepsilon \end{pmatrix}$, for $x \neq 0$, $\varepsilon \in \{\pm 1\}$

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Cent}(N_x^\varepsilon)$,

so $\gamma N_x^\varepsilon \gamma^{-1} = N_x^\varepsilon$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varepsilon & x \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \qquad (*)$$

$$= \begin{pmatrix} \varepsilon a & \varepsilon b + ax \\ \varepsilon c & \varepsilon d + cx \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \varepsilon - acx & a^2 x \\ -c^2 x & \varepsilon + acx \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon & x \\ 0 & \varepsilon \end{pmatrix}.$$

Hence $c=0$ and $a^2=1$, so $\mathrm{Cent}(N_x^\varepsilon)=N\cup(-N)$.

Similarly as in previous case, if $g=jN_x^\varepsilon j^{-1}$ $(x\neq 0)$, then

$$\sigma\in \mathrm{Cent}(g) \iff j^{-1}\sigma j\in N\cup(-N)$$
$$\iff \sigma\in N_g\cup(-N_g),$$
where $N_g=jNj^{-1}$.

iii) Check first that the normaliser of N is B.

Use $\circledast$ from ii) to check that if $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ is in the normaliser of N, then $c=0$.

Conclusion follows again by conjugation.

(3) • If $g=\pm Id$, easy check

The exercise follows from ex (1), since the trace is preserved in each conjugacy class, and each element $g\neq \pm Id_2$ is conjugate to a unique $\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix} \lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$

$x\neq 0$ $\quad\quad$ $\lambda\neq\pm 1$.

④ (i) Need to check that $(\gamma_1 \gamma_2).z = \gamma_1(\gamma_2 z)$,
$\forall\, \gamma_1, \gamma_2 \in SL_2(k)$, $z \in \mathbb{P}^1(k)$.

Can check this directly: if $\gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

$$\gamma_1.(\gamma_2 z) = \frac{a_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + b_1}{c_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + d_1} = \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)}$$

$$= (\gamma_1 \gamma_2).z.$$

(ii) By definition, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \frac{a}{c}$,

Hence $\mathrm{Stab}(\infty) = B$ $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \infty \iff c = 0 \right)$.

Now if $z = \sigma \infty$ (such $\sigma \in SL_2(k)$ always exists),

then
$$\gamma z = z \iff \gamma \sigma \infty = \sigma.\infty$$
$$\iff (\sigma^{-1} \gamma \sigma).\infty = \infty$$
$$\iff \sigma^{-1} \gamma \sigma \in B$$
$$\iff \gamma \in \sigma B \sigma^{-1} \text{ (a Borel group)}$$

Conclusion follows.

(iii) Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that $g\, z_1 = 0 \iff a z_1 + b = 0$

$g.z_2 = \infty \iff c z_2 + d = 0.$

Since $z_1, z_2 \in \mathbb{P}^1(k)$, we can always find $a, b, c, d \in k$ with this properties, and moreover can renormalize such that $\det(g) = 1$ (Note $\det(g) \neq 0$ since $z_1 \neq z_2$).

(iv) Note that $\mathrm{Stab}(\infty) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$

$$\mathrm{Stab}(0) = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\}$$

Hence $\mathrm{Stab}(\infty) \cap \mathrm{Stab}(0) = T$.

Now for general $z_1 \neq z_2$, $\exists ! g$ s.t. $z_1 = g\infty$

$z_2 = g0$.

Then $\mathrm{Stab}(z_1) = \mathrm{Stab}(g\infty) = g\, \mathrm{Stab}(\infty)\, g^{-1}$

$\mathrm{Stab}(z_2) = g\, \mathrm{Stab}(0)\, g^{-1}$

Hence $\mathrm{Stab}(z_1) \cap \mathrm{Stab}(z_2) = g(\mathrm{Stab}(\infty) \cap \mathrm{Stab}(0))g^{-1}$

$= gTg^{-1} =: T_g$ maximal torus

(v) From above, suffices to check for the point $0, \infty$ & z_1, any $z_1 \in \mathbb{P}^1(k)$, $0 \neq z_1 \neq \infty$.

There exist $\sigma \in SL_2(k)$ s.t. $z_1 = \sigma\infty$

Then $\mathrm{Stab}(0) \cap \mathrm{Stab}(\infty) \cap \mathrm{Stab}(z) = T \cap \sigma B \sigma^{-1}$

If $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, since $\sigma\infty \neq 0, \infty$, then $ac \neq 0$.

From exercise 2(i), in this case, we can check directly that $\sigma^{-1} T \sigma \cap B = \{\pm Id\}$.

(5) From previous exercise, we know that any Borel subgroup $B \subset SL_2(\bar{k})$ is the stabilizer of a point in $\mathbb{P}^1(\bar{k})$.

Also, all Borel subgroups are conjugate to each other (in $SL_2(\bar{k})$).

Let B Borel s.t. $B = Stab(z)$, $z \in \mathbb{P}^1(\bar{k})$.

Then for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B$, $\frac{az+b}{cz+d} = z$

$\Rightarrow az+b = cz^2 + dz$.

We need to show $\exists\, g \in SL_2(k)$ s.t. $gz \neq z$
(then $gBg^{-1} \neq B$, stabilizer of different points).

But if $g \in SL_2(k)$ with $gz = z$,
we have $\begin{cases} ad - bc = 1 \\ az + b - cz^2 - dz = 0. \end{cases}$

This will have $\ll |k|^2$ solutions in k^4.
But $|SL_2(k)| = k(k-1)^2 \gg |k|^3$.
We win if $|k|$ large enough.

6 (sketch)

- 6.24 follows directly from 6.23 + 6.20

For the rest, compare the proofs of 6.9 (LP for tori) & 6.23 (LP for tori, approx subgroup version), where the extra key input is 6.21 (the escape from Borel subgroups).

The same will hold if we redo proofs of 6.10, 6.11, 6.12, 6.14 respectively.

For $m>2$, apply 6.20 (for 6.25 & 6.28).